On Knowledge-based Fuzzy Sets

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Abstract

In fuzzy set, knowledge plays important roles in determining its membership function. By adding component of knowledge, this paper generalizes definition of fuzzy set based on probability theory. Some basic operations are re-defined. Granularity of knowledge is given in two frameworks, crisp granularity and fuzzy granularity. Objectivity and individuality mea- sure are proposed. Special attention is given to approximate reasoning in knowledge-based fuzzy sets representing fuzzy production rules as usually used in fuzzy expert system.

Keywords: Knowledge-based Fuzzy Set, Approximate Reasoning, Conditional Probability Relation, Objectivity Measure, Individuality Measure.

1. Introduction

There are at least two types of uncertainty, namely deterministic uncertainty called fuzziness and non-deterministic uncertainty called randomness.

In general, deterministic uncertainty may happen in the situation when one is subjectively able to determine or describe a given object, although somehow the object does not have a certain or clear definition. For example, a man describes a woman as a pretty woman. Obviously definition of a pretty woman is unclear, uncertain and subjective. The man however is convinced of what he describes as a pretty woman.

On the contrary, in non-deterministic uncertainty, one cannot determine or describe a given object even though the object has clear definition because human does not know what happen in the future and has limited knowledge. In other words, human is not omniscient being.

For example, in throwing a dice, even though there are six definable and certain possibilities of outcome, one however cannot assure the outcome of dice. Fuzzy set theory, proposed by Zadeh in 1965, is not to represent non deterministic situation of uncertainty such as randomness or stochastic process, but rather to represent deterministic uncertainty by a class or classes which do not possess sharply defined boundaries [11]. In deterministic uncertainty of fuzzy set, one may subjectively determine membership function of a given element by his knowledge. Different persons with different knowledge may provide different membership functions for elements in a universe with respect to a given fuzzy set. In other words, knowledge plays important roles in determining or defining a fuzzy set. Based on these reasons, we introduced knowledge-based representation of fuzzy set (knowledge-based fuzzy set, for short). Some basic concepts such as equality, containment, complementation, union and intersection are redefined [7]. In addition, by fuzzy conditional probability relation as proposed in [4,5], granularity of knowledge is given in two frameworks, crisp granularity and fuzzy granularity. By assuming each element of knowledge corresponds to a person and fuzzy set corresponds to problem or situation, we construct two asymmetric similarity classes of knowledge representing the granularity of knowledge based on the conditional probability relation. Also, on the assumption that the more a given description is acceptable by others, the more objective the description is, we may define objectivity measure in knowledge-based fuzzy sets. On the other hand, individuality measure is defined as the opposite of objectivity measure. Special attention is given to approximate reasoning in knowledge-based fuzzy sets representing fuzzy production rules as usually used in fuzzy expert system. It is proved that inference rules, which are similar to Armstrong's Axioms [1] for the fuzzy production rules, are both sound and complete.

2. Knowledge-based Fuzzy Set

As a quotation from Albert Einstein said "So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality.", we actually live in an uncertain world. At least, there are two types of uncertainty, namely deterministic uncertainty and non-deterministic uncertainty. Fuzzy set theory proposed by Zadeh is considered as an example of deterministic uncertainty. In deterministic uncertainty of fuzzy set, one may subjectively determine

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membership function of a given element by his knowledge. Different persons with different knowledge may provide different membership functions for elements in a universe with respect to a given fuzzy set. In other words, knowledge plays important roles in determining or defining a fuzzy set. In this section, a knowledge-based fuzzy sets is defined as follows.

Definition1: Let $U=\{u_1,\ldots,u_n\}$ be a set of elements and $K=\{k_1,\ldots,k_m\}$ be a set of knowledge. Then a fuzzy set A on U based on element of knowledge k_i denoted by $k_i(A)$ is defined a mapping from U to the closed interval [0,1] which is characterized by a membership function

$$\mu_{k_i(A)}:U\to[0,1]$$

We may then represent a given fuzzy set A on U in a fuzzy information table as shown in Table 1 in which $K(A)=\{k_1(A),\ldots,k_m(A)\}$ is set of knowledge-based fuzzy sets of A.

Table 1. Knowledge-based Fuzzy Set of A

$K(A)$	u_1	$\cdots$	u_n
$k_1(A)$	$\mu_{k_1(A)}(u_1)$	$\cdots$	$\mu_{k_1(A)}(u_n)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$k_m(A)$	$\mu_{k_m(A)}(u_1)$	$\cdots$	$\mu_{k_m(A)}(u_n)$

$\mu_{k_j(A)}(u_i)$ means membership function of element u_i in fuzzy set A based on element of knowledge k_j. An aggregation function [3,9] (f) may be applied in order to give a summary of all fuzzy sets. Formally, aggregation function on m fuzzy sets $(m\geq 2)$ is defined by

$$f:[0,1]^m\to[0,1].$$

When applied to the knowledge-based fuzzy sets as shown in Table I, function f produces a summary fuzzy set $S(A)$ by operating on the membership functions of all knowledge-based fuzzy sets for each $u\in U$ as follows:

$$\mu_{S(A)}(u)=f(\mu_{k_1(A)}(u),\ldots,\mu_{k_m(A)}(u)).$$

Here, depending on type of application, f might be defined by minimum, maximum, average, etc, in which f must satisfy,

$\min(x_1,\ldots,x_m)\leq f(x_1,\ldots,x_m)\leq\max(x_1,\ldots,x_m)$, for $x_j\in[0,1]$, $j\in N_m$.

By assuming that each elements of knowledge represents an opinion toward a given fuzzy set, we may need to consider other aggregation function such as,

$$f(x_1,\ldots,x_m)=\frac{\sum_j a_j x_j}{\sum_j a_j},\quad \forall j\in N_m,$$

where a_j be a coefficient corresponding to k_j, $a_j\in R^+$ and $x_j\in[0,1]$. Larger value of a_j denotes that k_j is more prominent in determining the summary fuzzy set.

There are several concepts relating to the knowledge-based fuzzy sets; for two fuzzy sets, $A,B\in F(U)$ on a set of elements U, where $F(U)$ is fuzzy power set of U and K is a set of knowledge,

Equality:

(e1)

$k_i(A)=k_i(B)\Leftrightarrow\mu_{k_i(A)}(u)=\mu_{k_j(B)}(u),\forall u\in U,$

(e2)

$A=B\Leftrightarrow\mu_{k(A)}(u)=\mu_{k(B)}(u),\forall u\in U,\forall k\in K,$

(e3)

$A\equiv B\Leftrightarrow\mu_{k_i(A)}(u)=\mu_{k_j(B)}(u),\forall u\in U,\forall k_i,k_j\in K,$

(e4)

$k_i=k_j\Leftrightarrow\mu_{k_i(A)}(u)=\mu_{k_j(A)}(u),\forall u\in U,\forall A\in F(U),$

(e5)

$k_i\sim k_j\Leftrightarrow\sum_u\mu_{k_i(A)}(u)=\sum_u\mu_{k_j(A)}(u),\forall A\in F(U).$

Containment:

(c1)

$k_i(A)\subset k_j(B)\Leftrightarrow\mu_{k_i(A)}(u)=\mu_{k_j(A)}(u),\forall u\in U,$

(c2)

$A\subset B\Leftrightarrow\mu_{k(A)}(u)\leq\mu_{k(B)}(u),\forall u\in U,\forall k\in K,$

(c3)

$A\Subset B\Leftrightarrow m_{k_i(A)}(u)\leq m_{k_j(B)}(u),\forall u\in U,\forall k_i,k_j\in K,$

(c4)

$k_i\prec k_j\Leftrightarrow\mu_{k_i(A)}(u)\leq\mu_{k_j(A)}(u),\forall u\in U,\forall A\in F(U),$

(c5)

$k_i\underset{\sim}{\prec}k_j\Leftrightarrow\sum_u\mu_{k_i(A)}(u)\leq\sum_u\mu_{k_j(A)}(u),\forall A\in F(U).$

Union:

(u1) $\mu_{k_j(A\cup B)}(u)=\max[\mu_{k_j(A)}(u),\mu_{k_j(B)}(u)],$

(u2) $\mu_{A\cup B}(u)=\max[\mu_{S(A)}(u),\mu_{S(B)}(u)].$

(u3) $\mu_{k_i(A)\cup k_j(B)}(u)=\max[\mu_{k_i(A)}(u),\mu_{k_j(B)}(u)],$

(u4) $\mu_{k_i\vee k_j(A)}(u)=\max[\mu_{k_i(A)}(u),\mu_{k_j(A)}(u)].$

Table 2. Knowledge-Based Fuzzy Set of Warm

$K(W)$	$20^\circ C$	$22^\circ C$	$24^\circ C$	$26^\circ C$	$28^\circ C$	$30^\circ C$	$32^\circ C$	$34^\circ C$	$36^\circ C$	$38^\circ C$	$40^\circ C$
$k_1(W)$	0	0	0.2	0.4	0.6	0.8	1.0	0.8	0.6	0.4	0.2
$k_2(W)$	0	0	0.3	0.6	1.0	0.6	0.3	0	0	0	0
$k_3(W)$	0.4	0.8	1.0	1.0	0.8	0.4	0	0	0	0	0
$k_4(W)$	0	0	0	0	0.5	1.0	1.0	0.5	0	0	0
$k_5(W)$	0.2	0.5	1.0	0.5	0.2	0	0	0	0	0	0
$k_6(W)$	0	0	0	0.6	1.0	1.0	0.6	0	0	0	0

Intersection:

(j1) $\mu_{k_j(A\cap B)}(u) = \min[\mu_{k_j(A)}(u), \mu_{k_j(B)}(u)]$,

(j2) $\mu_{(A\cap B)}(u) = \min[\mu_{f(A)}(u), \mu_{f(B)}(u)]$,

(j3) $\mu_{k_i(A)\cap k_j(B)}(u) = \min[\mu_{k_i(A)}(u), \mu_{k_j(B)}(u)]$,

(j4) $\mu_{k_i\wedge k_j(A)}(u) = \min[\mu_{k_i(A)}(u), \mu_{k_j(A)}(u)]$.

Note: related to union and intersection, we may consider an aggregation operation (*) as defined by:

$$\mu_{k_i(A)*k_j(A)}(u) = f(\mu_{k_i(A)}(u), \mu_{k_j(A)}(u)).$$

Complementation:

(n1) $\mu_{k_j(-A)}(u) = 1 - k_j^A(u)$,

(n2)

$$\mu_{-k_j(A)}(u) = \begin{cases} k_j^A(u), i \neq j, |K| = 2 \\ f(k_j^A(u), \ldots, k_{j-1}^A(u), k_{j+1}^A(u), \ldots, k_m^A(u), |K| = m\rangle 2 \end{cases}$$

where $k_j^A(u) = \mu_{k_j(A)}(u)$, for short.

From (n1) and (n2), we can get the following complementation:

$$\mu_{-k_j(-A)}(u) = \begin{cases} k_i^{-A}(u), i \neq j, |K| = 2 \\ f(k_1^{-A}(u), \ldots, k_{j-1}^{-A}(u), k_{j+1}^{-A}(u), \ldots, k_m^{-A}(u)) \\ , |K| = m\rangle 2, \end{cases}$$

where $k_j^{-A}(u) = 1 - \mu_{k_j(A)}(u)$, for short.

3. Granularity of Knowledge

Granularity of knowledge is proposed with intent to provide similarity classes of knowledge. In [4,5], fuzzy conditional probability relation was introduced as a more realistic relation in providing similarity between two elements or objects. Also in [5,6], two asymmetric similarity classes were introduced induced by fuzzy conditional probability relation. Corresponding to these two asymmetric similarity classes, we also propose two asymmetric similarity classes of knowledge. The concept of fuzzy conditional probability relations starts from definition of an interesting mathematical relation, weak fuzzy similarity relation as defined in the following definition.

Definition 2: A weak fuzzy similarity relation is a mapping, $s: F(U)\times F(U) \to [0,1]$, such that for $X, Y, Z \in F(U)$,

1. Reflexivity: $s(X, X) = 1$,

2. Conditional symmetry:
if $s(X,Y)\rangle 0$ then $s(Y,X)\rangle 0$

3. Conditional transitivity:
If $s(X,Y) \geq s(Y,X)\rangle 0$ and $s(Y,Z) \geq s(Z,Y)\rangle 0$ then $s(X,Z) \geq s(Z,X)$,

where U is an ordinary set of elements and $F(U)$ is fuzzy power sets of U.

Definition 3: A fuzzy conditional probability relation is a mapping, $R: F(U)\times F(U) \to [0,1]$, such that for $X, Y \in F(U)$,

$$R(X,Y) = P(X|Y) = P(Y \to X) = \frac{|X \cap Y|}{|Y|}$$

$$= \frac{\sum_{u\in U} \min\{\mu_X(u), \mu_Y(u)\}}{\sum_{u\in U} \mu_Y(u)},$$

where $R(X,Y)$ means the degree Y supports X or the degree Y is similar to X and $|Y| = \sum_{u\in U}\mu_Y(d)$ is regarded as cardinality of Y.

By definitions, a fuzzy conditional probability relation is considered as a concrete example of weak fuzzy similarity relations. In the definition of fuzzy conditional probability relation, the conditional probability of fuzzy sets (fuzzy events) is simply the relative cardinality expression by assuming that all the elements have equally probability or uniform probability distribution. We may consider fuzzy sets X and Y on $U = E^n$ (Euclidean n-space) which is characterized by membership function $\mu_X(e_1, \ldots, e_n)$ and $\mu_Y(e_1, \ldots, e_n)$, respectively with $u = (e_1, \ldots, e_n)$. Fuzzy conditional probability relation of fuzzy sets on E^n is given by

$$R(X,Y)=\frac{\sum_{e_1}\cdots\sum_{e_n}\min\{\mu_X(e_1,\ldots,e_n),\mu_Y(e_1,\ldots,e_n)\}}{\sum_{e_1}\cdots\sum_{e_n}\mu_Y(e_1,\ldots,e_n)}$$

Generally, fuzzy information table can be used to represent fuzzy sets. We however need fuzzy information (n+1)-dimensional table for representing fuzzy sets on E^n. Here, X and Y might be assumed as knowledge-based fuzzy sets in which each knowledge-based fuzzy set is regarded as fuzzy subset of elements in U. Furthermore, degree of similarity between two elements of knowledge, $k_i, k_j \in K$, in dealing a given fuzzy set A is provided by fuzzy conditional probability relation as shown in the following definition.

$$S_A(k_i,k_j)=R(k_i(A),k_j(A)),$$

where $k_i(A), k_j(A) \in K(A)$ are two knowledge-based fuzzy sets on U in dealing a given fuzzy set A.
R is fuzzy conditional probability relation. In this case, $S_A(k_i,k_j)$ means degree of k_j is similar to k_i in dealing a fuzzy label A in which $S_A(k_i,k_j)$ and $S_A(k_j,k_i)$ might have different values. For example, given a fuzzy information table of Warm (W) on degree Celsius as shown in Table 2. Degree of similarity between two elements knowledge, k_4 and k_6 for instance, can be calculated by

$$S_W(k_4,k_6)=\frac{0+0.5+1+0.6+0}{0.6+1+1+0.6}=\frac{2.1}{3.2},$$

$$S_W(k_6,k_4)=\frac{0+0.5+1+0.6+0}{0.5+1+1+0.5}=\frac{2.1}{3}.$$

It can be easily verified that degree of similarity between two elements of knowledge satisfies some properties such as for A is a given fuzzy set on U and $k_i, k_j, k_l \in K$ in which K is set of knowledge,

(r1) $[S_A(k_i,k_j)=S_A(k_j,k_i)=1, \forall A\in F(U)] \Leftrightarrow k_i = k_j$,

(r2) $[S_A(k_i,k_j)=1, S_A(k_j,k_i) < 1, \forall A\in F(U)] \Leftrightarrow k_j \prec k_i$,

(r3) $[S_A(k_i,k_j)=S_A(k_j,k_i) > 0, \forall A\in F(U)] \Rightarrow k_i \sim k_j$,

(r4) $[S_A(k_i,k_j) < S_A(k_j,k_i), \forall A\in F(U)] \Rightarrow k_i \precsim k_j$,

(r5) $S_A(k_j,k_i) > 0 \Leftrightarrow S_A(k_i,k_j) > 0$,

(r6) $[S_A(k_i,k_j)\ge S_A(k_j,k_i) > 0, S_A(k_j,k_l)\ge S_A(k_l,k_j) > 0] \Rightarrow S_A(k_i,k_l)\ge S_A(k_l,k_i)$

(r1) shows reflexive property in knowledge. In (r2), k_i covers k_j or k_j contains in k_i. As shown in (r3), k_i and k_j are the same tolerant (persons) in which we may consider cardinality of knowledge-based fuzzy set calculated by sum of membership function (see (e5) and (c5)) as measure of tolerance. On the other hand, k_j is more tolerant than k_i in (r4). Conditional symmetry and conditional transitivity are given in (r5) and (r6), respectively.
Based on degree of similarity between two elements of knowledge, we define two kinds of similarity classes of a given element of knowledge k_i.

Definition 4: Let K be a non-empty universe of knowledge, and S_A be degree of similarity between elements of knowledge in dealing a given fuzzy set A on a set of element U. For any element of knowledge $k_i \in K$, $S_A^\alpha(k_i)$ and $\rho_A^\alpha(k_i)$ are defined as the set of knowledge that supports k_i and the set supported by k_i, respectively by:

$$S_A^\alpha(k_i)=\{k\in K \mid S_A(k_i,k)\ge\alpha\},$$

$$\rho_A^\alpha(k_i)=\{k\in K \mid S_A(k,k_i)\ge\alpha\},$$

where $\alpha\in[0,1]$.
$S_A^\alpha(k_i)$ can also be interpreted as the set of knowledge that is similar to k_i. On the other hand, $\rho_A^\alpha(k_i)$ can be considered as the set of knowledge to which k_i is similar. Here, $S_A^\alpha(k_i)$ and $\rho_A^\alpha(k_i)$ are regarded as two different semantic interpretation of similarity classes in providing crisp granularity of knowledge. It can be proved that similarity class of knowledge satisfies some properties such as, if $k_i \precsim k_j$ then $S_A^\alpha(k_i)\subseteq S_A^\alpha(k_j)$, if $A\subseteq B$ then $S_A^\alpha(k)\subseteq S_B^\alpha(k)$ and if $k_i \sim k_j, \forall k_i,k_j\in K$ then $S_A^\alpha(k)=\rho_A^\alpha(k)$, $\forall k\in K, \forall A\in F(U)$. For two similarity classes of knowledge, $S_A^\alpha(k_i)$ and $S_A^\alpha(k_j)$, the complement, intersection and union are defined by:

$$-S_A^\alpha(k_i)=\{k\in K \mid k\notin S_A^\alpha(k_i)\}$$

$$S_A^\alpha(k_i)\cap S_A^\alpha(k_j)=\{k\in K \mid k\in S_A^\alpha(k_i) \quad and \quad k\in S_A^\alpha(k_j)\}$$

$$S_A^\alpha(k_i)\cup S_A^\alpha(k_j)=\{k\in K \mid k\in S_A^\alpha(k_i) \quad or \quad k\in S_A^\alpha(k_j)\}$$

Similarly, the complement, intersection and union can be defined for two similarity classes, $\rho_A^\alpha(k_i)$ and $\rho_A^\alpha(k_j)$. Obviously, the similarity classes of knowledge satisfy Boolean lattice, for the subsets are crisp sets. By the reflexivity, it follows that we can construct two crisp coverings of the universal set of knowledge,

$\{S^{\alpha}_{A}(k)|k\in K\}$ and $\{\rho^{\alpha}_{A}(k)|k\in K\}$. One may use these coverings of the universe to represent a generalization of rough sets as proposed in [5,6]. We can also withdraw $\alpha-cut$ or $\alpha-level$ set from Definition 4 with intent to provide a more generalization of similarity class. In this case, each similarity class is regarded as a fuzzy-granule as defined by

$$\mu_{S_A(k_i)}(k)=S_A(k_i,k),$$
$$\mu_{P_A(k_i)}(k)=S_A(k_i,k),\quad \forall k\in K,$$

where $\mu_{S_A(k_i)}(k)$ and $\mu_{\rho_A(k_i)}(k)$ are grades of membership of k in $S_A(k_i)$ and $\rho_A(k_i)$, respectively.

Similarly, $S_A(k)$ and $\rho_A(k)$ are regarded as fuzzy granularity of knowledge. Also, it can be proved that similarity class of knowledge satisfies some properties such as, if $k_i \preceq k_j$ then $S_A(k_i)\subseteq S_A(k_j)$, if $A\subseteq B$ then $S_A(k)\subseteq S_B(k)$ and if $k_i\sim k_j, \forall k_i,k_j\in K$ then $S_A(k)=\rho_A(k)$, $\forall k\in K, \forall A\in F(U)$. For two similarity classes of knowledge, $S_A(k_i)$ and $S_A(k_j)$, the complement, intersection and union are defined by:

$$\mu_{-S_A(k_i)}(k)=1-\mu_{-S_A(k_i)}(k),$$
$$\mu_{S_A(k_i)\cap S_A(k_j)}(k)=\min(\mu_{S_A(k_i)}(k),_{S_A(k_j)}(k)),$$
$$\mu_{S_A(k_i)\cup S_A(k_j)}(k)=\max(\mu_{S_A(k_i)}(k),_{S_A(k_j)}(k)).$$

Similarly, the complement, intersection and union can be defined for two similarity classes, $\rho_A(k_i)$ and $\rho_A(k_j)$. For similarity classes of knowledge in terms of fuzzy granularity are fuzzy sets, obviously some properties of Boolean lattice are not satisfied such as Law of contradiction and Law of excluded middle. Also, two different fuzzy coverings of the universal set of knowledge are given by $\{S_A(k)|k\in K\}$ and $\{\rho_A(k)|k\in K\}$ in which fuzzy granularity is a generalization of crisp granularity, implying that fuzzy covering is a generalization of crisp covering. Crisp and fuzzy granularity of knowledge play important role in representing classes (groups) of elements of knowledge (persons) who have similarities in dealing a given problem (situation), which is represented by a given fuzzy set.

Example 1: Given fuzzy information table of Warm (W) in Table 2. Two asymmetric similarity classes of k_4 and k_6 are given by:

$$S^{0.5}_{W}(k_4)=\{k_1,k_2,k_4,k_6\},$$
$$S^{0.5}_{W}(k_6)=\{k_2,k_4,k_6\},$$
$$\rho^{0.5}_{W}(k_4)=\{k_1,k_4,k_6\},$$
$$\rho^{0.5}_{W}(k_6)=\{k_1,k_2,k_3,k_4,k_6\}.$$

Two asymmetric similarity classes of k_4 and k_6 in fuzzy granularity are given by:

$$S_W(k_4)=\left\{\frac{0.5}{k_1},\frac{0.5}{k_2},\frac{0.2}{k_3},\frac{1.0}{k_4},\frac{0.6}{k_6}\right\},$$
$$S_W(k_6)=\left\{\frac{0.4}{k_1},\frac{0.8}{k_2},\frac{0.4}{k_3},\frac{0.7}{k_4},\frac{0.2}{k_5},\frac{1.0}{k_6}\right\},$$
$$\rho_W(k_4)=\left\{\frac{0.9}{k_1},\frac{0.4}{k_2},\frac{0.3}{k_3},\frac{1.0}{k_4},\frac{0.7}{k_6}\right\},$$
$$\rho_W(k_6)=\left\{\frac{0.7}{k_1},\frac{0.7}{k_2},\frac{0.5}{k_3},\frac{0.6}{k_4},\frac{0.2}{k_5},\frac{1.0}{k_6}\right\}$$

4. Objectivity and Individuality Measure

Generally, we may said that someone describes a given object objectively, if and only if his description is able to be accepted by all (persons). Actually, when we think of human being as personal being, then there are not objective description instead all description are subjective in the beginning. Therefore, standardization of definition has been made in order to avoid misunderstanding in communication such as quantitative measure in physics (measure of length, weight, time, energy, etc), regular shape of curves in geometry (line, circle, triangle, rectangle, etc), and so on. However, there is still a vast number of objects (anything) which cannot be defined objectively and acceptably by all (persons). Simply, on the assumption that the more a given description is acceptable by others, the more objective the description is, in this section, we define objectivity measure in knowledge-based fuzzy sets. First, objectivity measure is defined in terms of crisp granularity of knowledge as follows.

Definition 5: Let K be a non-empty universe of knowledge, and $\rho^{\alpha}_{A}(k_i)$ is set of knowledge that is supported by k_i. $\varphi^{\alpha}_{A}(k_i)$ is defined as degree of objectivity k_i in dealing fuzzy label A in the degree of similarity α by:

$$\varphi^{\alpha}_{A}(k_i)=\frac{|\rho^{\alpha}_{A}(k_i)|}{|K|},$$

where $\alpha\in[0,1]$.

On the other hand, we may define individuality measure as the opposite of objectivity measure by the following definition.

Definition 6: Let K be a non-empty universe of knowledge, and $\rho_A^\alpha(k_i)$ is set of knowledge that is supported by k_i. $\vartheta_A^\alpha(k_i)$ is defined as degree of individuality k_i in dealing fuzzy label A in the degree of similarity α by:

$$\vartheta_A^\alpha(k_i)=\frac{\left|K-\rho_A^\alpha(k_i)\right|+1}{|K|},$$

where $\alpha\in[0,1]$.

Obviously, relation between $\varphi_A^\alpha(k_i)$ and $\vartheta_A^\alpha(k_i)$ is given by:

(a) $\varphi_A^\alpha(k_i),\vartheta_A^\alpha(k_i)\in\left\{\frac{1}{|K|},\frac{2}{|K|},\ldots,1\right\},$

(b) $\varphi_A^\alpha(k_i)=1-\vartheta_A^\alpha(k_i)+\frac{1}{|K|}.$

(a) shows that minimum degrees of objectivity and individuality are $\frac{1}{|K|}$. By the reflexivity, it is obviously proved that at least there is an element of knowledge, k_i itself, who is perfectly supported by k_i. Even if all elements of K are supported by k_i, it does not mean that individuality of k_i become extinct. Here, degree of objectivity will be greater when there are more elements of knowledge, which are supported by k_i. On the other hand, degree of individuality will be greater when k_i is more unique. Obviously, $\varphi_A^\alpha(k_i)$ and $\vartheta_A^\alpha(k_i)$ are equal to 1 if and only if there is only one element in set of knowledge. In addition, degrees of objectivity and individuality rely on discrete value as a result of using crisp granularity. (b) shows a simple equation representing relation between $\varphi_A^\alpha(k_i)$ and $\vartheta_A^\alpha(k_i)$. It can be verified that $\varphi_A^\alpha(k_i)$ and $\vartheta_A^\alpha(k_i)$ satisfy some properties such as:

(i) $\varphi_A^1(k_i)=1,\vartheta_A^1(k_i)=\frac{1}{|K|}\Leftrightarrow k_i\preceq k,\forall k\in K,$

(ii) $\varphi_A^0(k)=1,\vartheta_A^0(k)=\frac{1}{|K|},\forall k\in K,$

(iii) $\alpha_1\le\alpha_2\Leftrightarrow\varphi_A^{\alpha_1}(k)\ge\varphi_A^{\alpha_2}(k),\vartheta_A^{\alpha_1}(k)\le\vartheta_A^{\alpha_2}(k),$ $\forall k\in K.$

From the set of K, we have a family of values $\{\varphi_A^\alpha(k)|k\in K\}$. To generalize all degrees of objectivity, we may consider the following three definitions:

(Minimum) $m j_A^a(K)=\min\{j_A^a(k)|k\in K\},$

(Maximum) $M j_A^a(K)=\max\{j_A^a(k)|k\in K\},$

(Average) $* j_A^a(K)=avg\{j_A^a(k)|k\in K\}$

In the same manner, from a family of value $\{\vartheta_A^\alpha(k)|k\in K\}$, we generalize all degrees of individuality by:

(Minimum) $m J_A^a(K)=\min\{J_A^a(k)|k\in K\},$

(Maximum) $M J_A^a(K)=\max\{J_A^a(k)|k\in K\},$

(Average) $* J_A^a(K)=avg\{J_A^a(k)|k\in K\}.$

By definition we can obtain some conclusions such as: $m\varphi_A^\alpha(K)=1\Leftrightarrow *\varphi_A^\alpha(K)=1$ means that objectivity of A is totality in degree of similarity α. Oppositely, $M\varphi_A^\alpha(K)=\frac{1}{|K|}\Leftrightarrow *\varphi_A^\alpha(K)=\frac{1}{|K|}$ means that objectivity of A is solitude in degree of similarity α. In the same manner, we also have $m\vartheta_A^\alpha(K)=1\Leftrightarrow *\vartheta_A^\alpha(K)=\frac{1}{|K|}$ and $M\vartheta_A^\alpha(K)=\frac{1}{|K|}\Leftrightarrow *\vartheta_A^\alpha(K)=\frac{1}{|K|}$ in terms of individuality of A. Related to the relation between $\varphi_A^\alpha(k_i)$ and $\vartheta_A^\alpha(k_i)$ in (b), we have $*\varphi_A^\alpha(K)=1-*\vartheta_A^\alpha(K)+\frac{1}{|K|}.$

Objectivity and individuality measure might be also defined in terms of fuzzy granularity of knowledge as shown in the following definition.

Definition 7: Let K be a non-empty universe of knowledge, and $\rho_A(k_i)$ is fuzzy set of knowledge that is supported by k_i. $\varphi_A(k_i)$ and $\vartheta_A(k_i)$ are defined as degree of objectivity and individuality of k_i, respectively, in dealing fuzzy label A as given by:

$$\varphi_A(k_i)=\frac{\sum_k\mu_{\rho_A(k_i)}(k)}{|K|},$$

$$\vartheta_A(k_i)=\frac{1+\sum_k(1-\mu_{\rho_A(k_i)}(k))}{|K|},\forall k\in K.$$

Also, relation between $\varphi_A(k_i)$ and $\vartheta_A(k_i)$ is given by:

(a) $\varphi_A(k_i),\vartheta_A(k_i)\in[\frac{1}{|K|},1],$

(b) $\varphi_A(k_i) = 1 - \vartheta_A(k_i) + \frac{1}{|K|}$.

Contrary to crisp granularity, in fuzzy granularity, degrees of objectivity and individuality rely on continuous value. It can be verified that $\varphi_A(k_i)$ and $\vartheta_A(k_i)$ satisfy

$$\varphi_A(k_i) = 1, \vartheta_A(k_i) = \frac{1}{|K|} \Leftrightarrow k_i \preceq k, \forall k \in K.$$

Similarly, from the set of K, we have two families of values $\{\varphi_A(k) | k \in K\}$ and $\{\vartheta_A(k) | k \in K\}$. Also, minimum, maximum and average functions can be used to generalize degrees of objectivity and individuality in the presence of fuzzy granularity of knowledge.

Related to Example 1, degrees of objectivity and individuality of k_4 and k_6 in terms of crisp granularity are given by:

$$\varphi_W^{0.5}(k_4) = \frac{6}{10}, \quad \vartheta_W^{0.5}(k_4) = \frac{3}{6},$$

$$\varphi_W^{0.5}(k_6) = \frac{7}{10}, \quad \vartheta_W^{0.5}(k_6) = \frac{4}{6}.$$

Degrees of objectivity and individuality of k_4 and k_6 in terms of fuzzy granularity are given by:

$$\varphi_W(k_4) = \frac{5.5}{10}, \quad \vartheta_W(k_4) = \frac{3.3}{6},$$

$$\varphi_W(k_6) = \frac{5.9}{10}, \quad \vartheta_W(k_4) = \frac{3.7}{6}.$$

5. Approximate Reasoning

Consider two persons, k_i and k_j for instance, argue about their different conclusions of a given premise. Let fuzzy label of A be a given premise and fuzzy label of B be the conclusion. We may represent relation between A and B by fuzzy production rules [9] which connect problems with solutions, antecedents with consequences, or premises with conclusions, as usually used in representing knowledge in fuzzy expert system. Generally, fuzzy production rules have the form as follows:

If A, then B,

where A,B are fuzzy sets.

Related to the knowledge-based fuzzy sets, conclusions of k_i and k_j are $k_i(B)$ and $k_j(B)$ in which $k_i(B) \neq k_j(B)$. Our problem is to determine which one has the right conclusion - k_i or k_j. Sometimes, different views or understanding of the given premise are the cause of different conclusions. Here, we may summarize their relations into four possibilities:

1. $k_i(A) = k_j(A), k_i(B) = k_j(B)$, there is no problem because both k_i and k_j are exactly the same in understanding the given premise and giving the conclusion.
2. $k_i(A) = k_j(A), k_i(B) \neq k_j(B)$, both k_i and k_j have the same understanding of the given premise, but they have different conclusions; that is the problem.
3. $k_i(A) \neq k_j(A), k_i(B) = k_j(B)$, because both k_i and k_j have different understanding of the premise even though they have the same conclusion, their conclusions should be treated independently.
4. $k_i(A) \neq k_j(A), k_i(B) \neq k_j(B)$, with the same as point 3, their conclusions are independent so that their different conclusions are able to be understood and tolerated.

From the four possibilities, we have a problem in point 2. Fairly, k_i and k_j have the same degree of correctness; let say 0.5 in probability measure. By granularity of knowledge as proposed in the previous section, degree of correctness of approximate reasoning in fuzzy production rule by certain knowledge (person) can be approximately calculated as follows.

Definition 8: Let K be a non-empty universe of knowledge, and $S_A^\alpha(k), S_B^\alpha(k)$ be crisp granularity of knowledge of $k \in K$ in dealing fuzzy label A and fuzzy label B, respectively. $\delta_\alpha(A \xrightarrow{k} B)$ is defined as degree of correctness of approximate reasoning k in representing conclusion of B as given premise of A by

$$\delta_\alpha(A \xrightarrow{k} B) = \frac{|S_A^\alpha(k) \cap S_B^\alpha(k)|}{S_A^\alpha(k)},$$

where $\alpha \in [0,1]$ and $|.|$ be a cardinality of set.

From the set of K, we have a family of values $\{\delta_\alpha(A \xrightarrow{k} B) | k \in K\}$. To generalize all degrees of correctness, we may consider the following three definitions:

Minimum: $d_a^m(A \xrightarrow{k} B) = \min\{d_a(A \xrightarrow{k} B) | k \in K\}$,

Maximum: $d_a^M(A \xrightarrow{k} B) = \max\{d_a(A \xrightarrow{k} B) | k \in K\}$

Average: $d_a^*(A \xrightarrow{k} B) = avg\{d_a(A \xrightarrow{k} B) | k \in K\}$

By definition we can obtain some properties such as: if connection between premise A and conclusion B is totally correct then $S_A^\alpha(k) \subseteq S_B^\alpha(k)$ for all $k \in K$. It also means that the similarity classes of knowledge in dealing fuzzy label A is finer than the similarity classes of knowledge in dealing fuzzy label B.

$\delta_\alpha^m(A\xrightarrow{k}B)=1 \Leftrightarrow \delta_\alpha^*(A\xrightarrow{k}B)=1$, similarly $\delta_\alpha^m(A\xrightarrow{k}B)\langle 1 \Leftrightarrow \delta_\alpha^*(A\xrightarrow{k}B)\langle 1$. If $\delta_\alpha^*(A\xrightarrow{k}B)=1$ then we say that B is general conclusion as given premise A, otherwise if $\delta_\alpha^*(A\xrightarrow{k}B)\langle 1$, we say that B is partial conclusion (in a degree $\delta_\alpha^m(A\xrightarrow{k}B)$) as given premise A. Likewise, if $\delta_\alpha(A\xrightarrow{k}B)=1$ we say that B is general conclusion as given premise A in an element of knowledge $k\in K$, otherwise if $\delta_\alpha(A\xrightarrow{k}B)\langle 1$, we say that B is partial conclusion (in a degree $\delta_\alpha(A\xrightarrow{k}B)$) as given premise A in element of knowledge k.

Degree of correctness in Definition 5 may also be generalized and calculated by fuzzy granularity of knowledge as defined by:

$$\delta_\alpha(A\xrightarrow{k_i}B)=\frac{\sum_{k\in K}\min\{\mu_{S_A(k_i)}(k),\mu_{S_B(k_i)}(k)\}}{\sum_{k\in K}\mu_{S_A(k_i)}(k)},$$

where intersection is defined as minimum and cardinality is provided by sum of membership function. Similarly, $\delta^m(A\xrightarrow{K}B),\delta^M(A\xrightarrow{K}B)$ and $\delta^*(A\xrightarrow{K}B)$ may also be defined to generalize all degrees of correctness.

Also, it depends on the application in which for example all of us agree that there is a causal relationship between A and B. However the relationship might be unclear in determining which one is premise and which one is conclusion.

For example, let $K=\{k_1,k_2,k_3\}$ be set of knowledge. Interpretation of fuzzy labels A and B based on K is arbitrarily given in the following figure.

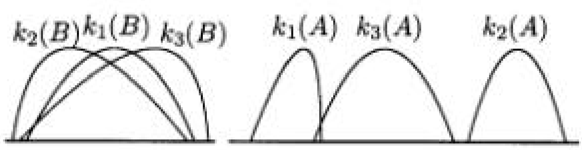

Figure 1. ?

Obviously, the figure shows that all elements of K have almost the same interpretation of B, but they have different interpretation of A. If we consider B as premise and A as conclusion, a problem arises in determining what interpretation of A should be used as conclusion in the case of almost the same interpretation of B. On the other hand, if we consider A as premise and B as conclusion, no matter in the beginning they have different interpretation of A as premise, finally they will have the same conclusion (in α level set). Therefore, we should consider A as premise and B as conclusion in the given example. In this case, by the granularity of knowledge, we are able to determine which one should be a premise and which one should be a conclusion in the causal relationship between A and B. Here, similarity classes of knowledge in dealing premise should be finer than similarity classes of knowledge in dealing conclusion. Fuzzy production rule represents A as premise and B as conclusion (A determines B, for short) in element of knowledge $k\in K$ defined by:

(a) $A\xrightarrow{k}B \Leftrightarrow \delta_\alpha(B\xrightarrow{k}A)\langle\delta_\alpha(A\xrightarrow{k}B)=1$, (A strongly determines B)

(b) $A\overset{k}{\rightsquigarrow}B \Longleftrightarrow \delta_\alpha(B\xrightarrow{k}A)<\delta_\alpha(A\xrightarrow{k}B)<1$, (A weakly determines B)

(c) $A\xleftrightarrow{k}B \Leftrightarrow \delta_\alpha(A\xrightarrow{k}B)=\delta_\alpha(B\xrightarrow{k}A)=1$, (A strongly equals B)

(d) $A\overset{k}{\leftrightsquigarrow}B \Longleftrightarrow \delta_\alpha(A\xrightarrow{k}B)=\delta_\alpha(B\xrightarrow{k}A)<1$, (A weakly equals B)

where K is set of knowledge. $\delta_\alpha(A\xrightarrow{k}B)$ and $\delta_\alpha(B\xrightarrow{k}A)$ can be generalized and changed to $\delta(A\xrightarrow{k}B)$ and $\delta(B\xrightarrow{k}A)$, respectively. It is also necessary to consider and define some sets of fuzzy production rules which are subsets of K as defined by

(i) $\overset{A\rightarrow B}{K}=\left\{k\in K \middle| A\xrightarrow{k}B\right\}$, (set of A strongly determines B)

(ii) $\overset{A\rightsquigarrow B}{K}=\{k\in K|A\overset{k}{\rightsquigarrow}B\}$ (set of A weakly determines B)

(iii) $\overset{A\leftrightarrow B}{K}=\left\{k\in K \middle| A\xleftrightarrow{k}B\right\}$, (set of A strongly determines B)

(iv) $\overset{A\leftrightsquigarrow B}{K}=\{k\in K|A\overset{k}{\leftrightsquigarrow}B\}$, (set of A weakly equals B)

where they are satisfied:

$\overset{A\rightarrow B}{K}\cup\overset{A\rightsquigarrow B}{K}\cup\overset{A\leftrightarrow B}{K}\cup\overset{A\leftrightsquigarrow B}{K}\cup\overset{B\rightarrow A}{K}\cup\overset{B\rightsquigarrow A}{K}=K$, where $\overset{A\rightarrow B}{K}$, $\overset{A\rightsquigarrow B}{K}$ $\overset{A\leftrightarrow B}{K}$, $\overset{A\leftrightsquigarrow B}{K}$ $\overset{B\rightarrow A}{K}$, and $\overset{B\rightsquigarrow A}{K}$ are disjoint subsets of K. In order to generalize fuzzy production rules, we need to calculate cardinality of the sets of fuzzy production rule, A determines B, by

$$\mathcal{C}(A\rightarrow B)=|\overset{A\rightarrow B}{K}|+0.75\times|\overset{A\rightsquigarrow B}{K}|+0.5\times|\overset{A\leftrightsquigarrow B}{K}|+0.25\times|\overset{B\rightsquigarrow A}{K}|.$$

Similarly, we have

$$\mathcal{C}(B\rightarrow A)=|\overset{B\rightarrow A}{K}|+0.75\times|\overset{B\rightsquigarrow A}{K}|+0.5\times|\overset{B\leftrightsquigarrow A}{K}|+0.25\times|\overset{A\rightsquigarrow B}{K}|.$$

, where |.| means cardinality of set. $|\overset{A\leftrightarrow B}{K}|$ is not included in calculating cardinality of set of A determines B with intent to deal $A \leftrightarrow B$ as a special condition. Simply, coefficients of cardinality of sets are given with intervals of 0.25 because there are four sets of fuzzy production rules that involve in the calculation. We then define fuzzy production rules in K as follows.

(Rule 1) $A \rightarrow B \Leftrightarrow C(A \rightarrow B) = |K|,$

(Rule 2) $A \leftrightarrow B \Leftrightarrow C(A \rightarrow B) = C(B \rightarrow A) = 0,$

(Rule 3) $A \rightsquigarrow B \Longleftrightarrow |K| > C(A \rightarrow B) > C(B \rightarrow A),$

(Rule 4) $A \leadsto B \Longleftrightarrow C(A \rightarrow B) = C(B \rightarrow A) > 0,$

where |.|s is cardinality of set. It can be also said that $A \rightarrow B \Leftrightarrow \overset{A\rightarrow B}{K} = K$ and $A \leftrightarrow B \Leftrightarrow \overset{A\leftrightarrow B}{K} = K$. It can be proved that the fuzzy production rules satisfies Armstrong's Axioms [1], such that for A,B,C are fuzzy

Reflexivity: $A \subseteq B \Rightarrow A \rightarrow B, \quad A, B \in F(U),$

Augmentation: $A \rightarrow B \Rightarrow (A \cap C) \rightarrow B, A, C \in F(U),$

Transitivity: ($A \rightarrow B$ and $B \rightarrow C) \Rightarrow A \rightarrow C$.

6. Conclusions

We proposed knowledge-based representation of fuzzy sets. Aggregation function was used to provide a summary fuzzy set. Some basic concepts and operations such as equality, containment, union, intersection and complementation were defined in terms of knowledge-based fuzzy sets. Granularity of knowledge was proposed in two frameworks, crisp granularity and fuzzy granularity, in order to provide two asymmetric similarity classes of knowledge in dealing a fuzzy set representing a problem. On the assumption that the more a given description is acceptable by others, the more objective the description is, we defined objectivity measure in knowledge-based fuzzy sets. Contrary to objectivity measure, we also proposed individuality measure. Special attention was given to approximate reasoning of knowledge-based fuzzy sets in representing fuzzy production rules. Inference rules, which are similar to Armstrong's Axioms for the fuzzy production rules, are both sound and complete.

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